Calculations for an Oblate Spheroid Neutron Star
(in connection with Figure 27)

See "How the rotational momentum is accommodated" in Chapter 6 of
The Nature of Gravitational Collapse

Conrad Ranzan

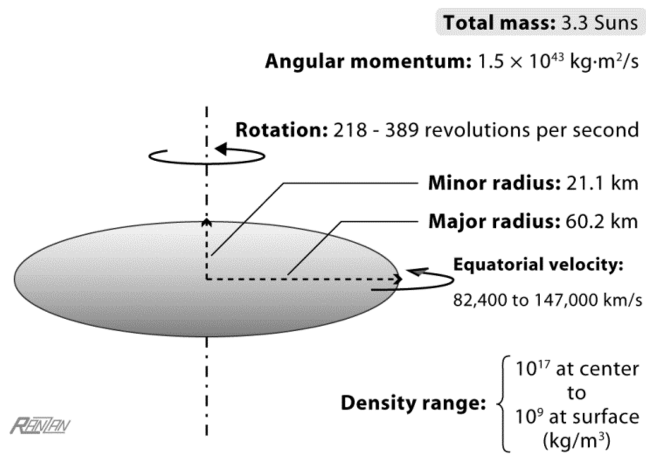

Fig. 27. High-spin neutron star model: It conforms to the laws of physics and fulfills the requirements described in the text, namely, contained mass, conserved angular momentum, linear density range, rotation rate, and special-relativity conformance. The spin rate of the neutron structure will range from about 218 to 389 revolutions per second depending on whether the structure is treated as a "rigid" body (in which case the angular velocity ω is uniform for all layers), or whether it is treated as an onion-like body of frictionless concentric shells (in which case the equatorial linear velocity is the same for all layers, but ω is different for each layer).

The purpose of this Supplement is to find the characteristics of a neutron star given only its mass of **3.3 Solar masses** (or 6.57×10^{30} kg) and its **angular momentum (1.5×10^{43} kg·m/s)**.

Derivation of Mass expression in terms of radii and density

Cross-section of structure:

The cross-section may be approximated as an ellipse whose equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{or}\ \frac{r^2}{a^2}+\frac{h^2}{b^2}=1.$$

Solving for height, h, gives: $h=\pm\frac{b}{a}\sqrt{a^2-r^2}$.

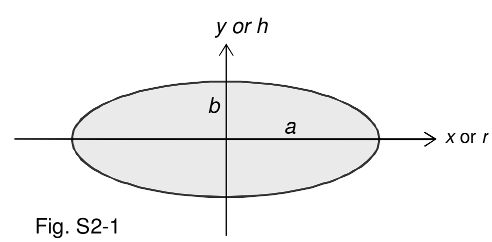

Fig. S2-1

Volume of elemental cylinder:

$$\mathrm{Vol}_{\mathrm{cylinder}} = \text{circumference} \times \text{height} \times \text{thickness}$$

$$= 2\pi r\left(2\frac{b}{a}\sqrt{a^2-r^2}\right)dr\,.$$

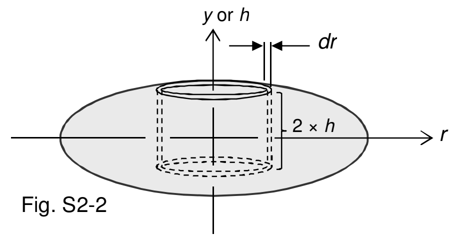

Fig. S2-2

Density function:

It is assumed that the density ranges linearly from ρ_c, at the core center, to ρ_s at the surface.

Average density of a cylinder is ½($\rho_{\max@r} + \rho_s$);

$$\rho_{\max}(r) = \left(\frac{\rho_s-\rho_c}{a}\right)r + \rho_c\,,$$ this is just the line in the density graph;

Average density, as a function of r:.

$$\rho_{\mathrm{ave}}(r) = \frac{1}{2}\left[\left(\frac{\rho_s-\rho_c}{a}\right)r + \rho_c\right].$$

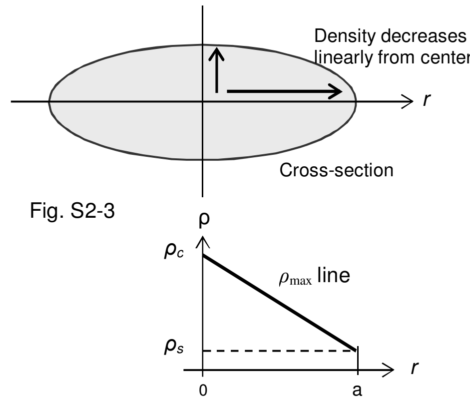

Fig. S2-3

Mass of elemental cylinder:

$$dM = (\text{Vol @ } r) \times (\text{Ave density @ } r)\,;$$

$$= \left(2\pi r\left(2\frac{b}{a}\sqrt{a^2-r^2}\right)dr\right)\left[\left(\frac{\rho_s-\rho_c}{2a}\right)r + \frac{\rho_c}{2}\right];$$

$$dM = 4\pi\frac{b}{a}\left[\left(\frac{\rho_s-\rho_c}{2a}\right)r^2\sqrt{a^2-r^2}\,dr + \left(\frac{\rho_c}{2}\right)r\sqrt{a^2-r^2}\,dr\right].$$

Mass of oblate spheroid (general equation):

$$M = 4\pi\frac{b}{a}\left[\left(\frac{\rho_s-\rho_c}{2a}\right)\underbrace{\int_0^a r^2\sqrt{a^2-r^2}\,dr}_{\text{Solve with Integral \#30.}} + \left(\frac{\rho_c}{2}\right)\underbrace{\int_0^a r\sqrt{a^2-r^2}\,dr}_{\text{Solve by substitution.}}\right];$$

$$= 4\pi\frac{b}{a}\left[\left(\frac{\rho_s-\rho_c}{2a}\right)\left(1.57\frac{a^4}{8}\right) + \left(\frac{\rho_c}{2}\right)\frac{a^3}{3}\right];$$

…

$$= \underbrace{4\pi ba^2}_{\text{length units}}\underbrace{\left[0.09812(\rho_s-\rho_c) + 0.1666(\rho_c)\right]}_{\text{density units}}$$

Applying Integral # 30 from *Calculus and Analytic Geometry, 4th Ed.*, George B. Thomas (Addison-Wesley, Reading, Massachusetts, 1968):

$$\int_0^a r^2\sqrt{a^2-r^2}\,dr = \frac{a^4}{8}\sin^{-1}\frac{r}{a} - \frac{1}{8}r\sqrt{a^2-r^2}\left(a^2-2r^2\right)\Bigg|_0^a,$$

$$= \frac{a^4}{8}\sin^{-1}\frac{a}{a} - \frac{1}{8}a\sqrt{a^2-a^2}\left(a^2-2a^2\right) - \frac{a^4}{8}\sin^{-1}\frac{0}{a} + \frac{1}{8}(0)$$

$$= \frac{a^4}{8}(1.57) - 0 - 0 + 0\ .$$

Solving by substitution:

Since $\dfrac{d\left(a^2-r^2\right)}{dr} = -2r\,,$

$$\int_0^a r\sqrt{a^2-r^2}\,dr = -\frac{1}{3}\left(a^2-r^2\right)^{3/2}\Bigg|_0^a,$$

$$= -\frac{1}{3}\left(a^2-a^2\right)^{3/2} + \frac{1}{3}\left(a^2-0^2\right)^{3/2},$$

$$= \frac{a^3}{3}.$$

Calculation of minor and major radii:

The oblateness of the neutron star depends on the centrifugal effect, which, in turn, depends on an intricate combination of the mass distribution, the rate of spin, and the frame-dragging effect. The *frame dragging* is a measure of the intensity of the aether's spiral motion.

The complexity here lies in the fact that the star is rotating partially THROUGH aether (the *vacuum*) and partially WITH aether.

To overcome the complexity of the inter-relationship and circumvent the calculation of its net effect on oblateness, a reasonable assumption is made. The assumption, for the case under discussion, is that the ratio of semi-minor and semi-major axes is 1.4 to 4.0. This means

$\frac{b}{a} = \frac{1.4}{4} = 0.35$ or $\boldsymbol{b = 0.35\,a}$.

Now for the density term: Using the generally accepted values for neutron star density ($\rho_s = 10^9$ kg/m^3 and $\rho_c = 10^{17}$ kg/m^3), the density term in the Mass equation, above, evaluates to **0.06854×10^{17} kg/m^3**.

Substitute these values (and the given mass) into the Mass equation and solve for the main radius (the semi-major axis):

$$M_{3.3\text{Sun}} = 4\pi b a^2 \left[0.09812\left(\rho_s - \rho_c\right) + 0.16666\left(\rho_c\right)\right];$$

$$6.57\times10^{30}\,kg = 4\pi(0.35a)a^2\left[0.06854\times10^{17}\,kg\,/\,m^3\right];$$

$$a = 6.018\times10^4 \text{ meters.}$$

Thus, the radius in the plane of rotation is **60.2 kilometers**.

And the radius coinciding with the axis of rotation is (0.35 × 60.2 km) **21.1 kilometers**.

Derivation of angular momentum expression (assuming constant rotation speed υ_{const})

The assumed condition here is that the rotation speed of each mass element is the same for all distances from the axis of rotation. In other words, the neutron star is treated as an onion-like body of frictionless concentric shells, each rotating with the same equatorial speed (but different rotation periods).

Angular momentum of elemental cylinder/ring:

Consider the mass element shown in Figure S2-2.

Start the derivation with the textbook fact:

(Angular momentum) = (moment of inertia) × (angular velocity) ;

And express it in differential form:

$dL = dI \times \omega$, where the moment of inertia of a cylinder is I = (mass)(radius)^2;

$dL = dM \times r^2 \times \upsilon_{\text{const}}/r$;

$\quad = dM \times r \times \upsilon_{\text{const}}$;

$$= 4\pi\frac{b}{a}\left[\left(\frac{\rho_s-\rho_c}{2a}\right)r^3\sqrt{a^2-r^2}\,dr + \left(\frac{\rho_c}{2}\right)r^2\sqrt{a^2-r^2}\,dr\right]\upsilon_{\text{const}}.$$

Angular momentum of oblate neutron star (general equation):

$$L = 4\pi\frac{b}{a}\left[\left(\frac{\rho_s-\rho_c}{2a}\right)\underbrace{\int_0^a r^3\sqrt{a^2-r^2}\,dr}_{} + \left(\frac{\rho_c}{2}\right)\underbrace{\int_0^a r^2\sqrt{a^2-r^2}\,dr}_{}\right]\upsilon_{\text{const}};$$

Solve by substituting $z = (a^2 - r^2)^{1/2}$.

Solve with Integral #30 (see textbox above).

$$= 4\pi \frac{b}{a}\left[\frac{(\rho_{\mathrm{s}}-\rho_{\mathrm{c}})}{2a} \left| \frac{(a^2-r^2)^{5/2}}{5} - \frac{a^2(a^2-r^2)^{3/2}}{3} \right|_0^a + \frac{\rho_{\mathrm{c}}}{2} \left| \frac{a^4}{8}\sin^{-1}\frac{r}{a} - \frac{r}{8}\sqrt{a^2-r^2}\,(a^2-2r^2) \right|_0^a \right] \upsilon_{\mathrm{const}} ;$$

$$= 4\pi \frac{b}{a}\left[\frac{(\rho_{\mathrm{s}}-\rho_{\mathrm{c}})}{2a}\left(\frac{2a^5}{15}\right) + \frac{\rho_{\mathrm{c}}}{2}\left(\frac{a^4}{8}1.57\right) \right] \upsilon_{\mathrm{const}} ;$$

…

$$= 4\pi b a^3 \underbrace{\left[0.06666(\rho_{\mathrm{s}}-\rho_{\mathrm{c}}) + 0.09812(\rho_{\mathrm{c}}) \right]}_{\text{density term}} \upsilon_{\mathrm{const}} .$$

With the generally accepted values for neutron star density ($\rho_{\mathrm{s}} = 10^9$ kg/m^3 and $\rho_{\mathrm{c}} = 10^{17}$ kg/m^3), the density term takes the value **0.03146×10^{17} kg/m^3**.

Calculating the rotation rate:

Rearranging the above equation gives:

$$\upsilon_{\mathrm{const}} = \frac{(\text{Angular momentum})}{4\pi b a^3\,(\text{Density term})\left(0.03146\times10^{17}\right)} ;$$

$$\upsilon_{\mathrm{const}} = \frac{\left(1.5\times10^{43}\,kg\cdot m^2/s\right)}{4\pi\left(21.1\times10^3 m\right)\left(60.2\times10^3 m\right)^3\left(0.03146\times10^{17}\,kg/m^3\right)} ;$$

$\upsilon_{\mathrm{const}} = 0.824\times10^8\,m/s$ or **82,400 kilometers per second**.

This is the speed of the mass at the equator, a speed that corresponds to 27.5 percent of lightspeed.

It means the neutron star, if treated as an onion-like body of frictionless concentric shells (in which case the equatorial linear velocity is the same for all layers, while the angular rotation ω is different for each layer), is expected to rotate at **218 times per second**.

The above approach considered the neutron star to be a *non*-rigid body. The following derivation treats it as a rigid body.

Derivation of angular momentum expression (assuming uniform angular velocity ω)

The assumed condition here is that the angular velocity ω is the same for all parts of the structure. In other words, the neutron star is rotating as a *rigid* body. (The rotation period is the same for all elements of the structure.)

Moment of Inertia of elemental cylinder/ring:

Moment of Inertia of the elemental mass cylinder shown in Fig. S2-2 is

$dI = (\text{mass})\times(\text{radius})^2$;

$= dM \times r^2$;

$$= 4\pi \frac{b}{a}\left[\left(\frac{\rho_{\mathrm{s}}-\rho_{\mathrm{c}}}{2a}\right) r^4\sqrt{a^2-r^2}\,dr + \left(\frac{\rho_{\mathrm{c}}}{2}\right) r^3\sqrt{a^2-r^2}\,dr \right]$$

Moment of Inertia of rigid oblate neutron star (general equation):

$$I = 4\pi \frac{b}{a}\left[\left(\frac{\rho_s - \rho_c}{2a}\right)\underbrace{\int_0^a r^4\sqrt{a^2 - r^2}\,dr}_{} + \left(\frac{\rho_c}{2}\right)\underbrace{\int_0^a r^3\sqrt{a^2 - r^2}\,dr}_{}\right]$$

Solve by substituting and with Integral #51 (see adjacent textbox).

Solve by substituting $z = (a^2 - r^2)^{1/2}$.

$$= 4\pi \frac{b}{a}\left[\begin{array}{l} \dfrac{(\rho_s - \rho_c)}{2a}\dfrac{1}{2}\left|\dfrac{\left(r^2 + \frac{1}{2}a^2\right)\left(2r^2 - \frac{3}{2}a^2\right)\sqrt{a^2r^2 - r^4}}{6} + \dfrac{a^6}{8}\sin^{-1}\left(\dfrac{r^2 - \frac{1}{2}a^2}{\frac{1}{2}a^2}\right)\right|_0^a \\ + \dfrac{\rho_c}{2}\left|\dfrac{\left(a^2 - r^2\right)^{5/2}}{5} - \dfrac{a^2\left(a^2 - r^2\right)^{3/2}}{3}\right|_0^a \end{array}\right];$$

…

$$= 4\pi \frac{b}{a}\left[\frac{(\rho_s - \rho_c)}{2a}\left(\frac{1.57a^6}{16}\right) + \frac{\rho_c}{2}\left(\frac{2a^5}{15}\right)\right];$$

$$= 4\pi b a^4\left[0.04906(\rho_s - \rho_c) + 0.06666\rho_c\right]$$

Integral # 51 from *Calculus and Analytic Geometry, 4th Ed.*, George B. Thomas (Addison-Wesley, Reading, Massachusetts, 1968):

$$\int x\sqrt{2ax - x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax - x^2}}{6} + \frac{a^3}{2}\sin^{-1}\frac{x-a}{a} + C.$$

Angular momentum of rigid oblate neutron star (general equation):

(Rotational momentum) = (moment of inertia) × (angular velocity),

$L = I \times \omega$;

$= 4\pi b a^4\left[0.04906(\rho_s - \rho_c) + 0.06666\rho_c\right]\omega$.

Substitute the given and calculated values to obtain the angular momentum as follows:

$$\omega = \frac{L}{4\pi b a^4\,(\text{density term})};$$

$$\omega = \frac{1.5\times10^{43}\,kg\cdot m^2/s}{4\pi\left(21.1\times10^3 m\right)\left(60.2\times10^3 m\right)^4\left(0.01761\times10^{17}\,kg/m^3\right)};$$

ω = 2446 radians per second, which corresponds to **389 revolutions per second**.

In this case, the rotational speed at the equator would be 147,100 kilometers per second or **49.0 percent of lightspeed**.

Conclusion

For the 10-Solar-mass star discussed in the book, the collapsed neutron structure will have a rotation rate somewhere between the two extremes of 218 and 389 revolutions per second.

(Note that a reduction of the centrifugal effect due to aether vortex motion has relevance here, but is not explicitly included in the calculations. Because of the induced aether vortex, the structure has a reduced tendency of centrifugally flying apart.)

* * *